JACOB'S LADDERS AND THE TANGENT LAW FOR SHORT PARTS OF THE HARDY-LITTLEWOOD INTEGRAL

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Abstract. The elementary geometric properties of the Jacob's ladders [7] lead to a class of new formulae for short parts of the Hardy-Littlewood integral. This class of formulae cannot be obtained by methods of Balasubramanian, Heath-Brown and Ivic.

1. Necessity of a new expression for short parts of the Hardy-Littlewood integral

1.1. Titchmarsh-Kober-Atkinson (TKA) formula (see [3], p. 141)

$$\int_0^\infty Z^2(t)e^{-2\delta t}\mathrm{d}t=\frac{c-\ln(4\pi\delta)}{2\sin(\delta)}+\sum_{n=0}^N c_n\delta^n+\mathcal{O}(\delta^{N+1}) \tag{1.1}$$

remained as an isolated result for the period of 56 years. We have discovered [7] the nonlinear integral equation

$$\int_0^{\mu[x(T)]} Z^2(t)e^{-\frac{2}{x(T)}t}\mathrm{d}t=\int_0^T Z^2(t)\mathrm{d}t, \tag{1.2}$$

in which the essence of the TKA formula is encoded. Namely, we have shown in [7] that the following almost-exact expression for the Hardy-Littlewood integral

$$\int_0^T Z^2(t)\mathrm{d}t=\frac{\varphi(T)}{2}\ln\frac{\varphi(T)}{2}+(c-\ln(2\pi))\frac{\varphi(T)}{2}+c_0+\mathcal{O}\left(\frac{\ln T}{T}\right) \tag{1.3}$$

takes place, where $\varphi(T)$ is the Jacob's ladder, i.e. an arbitrary solution to the nonlinear integral equation (1.2).

Remark 1. Let us remind that

(A) The Good's Ω-theorem implies for the Balasubramanian's formula [1]

$$\int_0^T Z^2(t)\mathrm{d}t=T\ln T+(2c-1-\ln 2\pi)T+R(T),\quad R(T)=\mathcal{O}(T^{1/3+\epsilon}) \tag{1.4}$$

that

$$\limsup_{T\to+\infty}|R(T)|=+\infty,$$

i.e. the error term in (1.4) is unbounded at $T\to\infty$.

(B) In the case of our result (1.3) the error term tends to zero as T goes to infinity, namely,

$$\lim_{T\to\infty}|r(T)|=0,\quad r(T)=\mathcal{O}\left(\frac{\ln T}{T}\right),$$

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i.e. our formula is almost exact (see [7]).

1.2. The Balasubramanian's formula implies, for $U_0 = T^{1/3+2\epsilon}$,

$$\int_T^{T+U_0} Z^2 t\mathrm{d}t = U_0 \ln T + (2c - \ln 2\pi)U_0 + \mathcal{O}(T^{1/3+\epsilon}). \tag{1.5}$$

Furthermore let us remind the Heath-Brown's estimate (see [5], (7.20) p. 178)

(1.6)
$$\int_{T-G}^{T+G} Z^2(t)\mathrm{d}t = \mathcal{O}\left\{ G\ln T + G\sum_K (TK)^{-1/4}\left(|S(K)| + K^{-1}\int_0^K |S(x)|\mathrm{d}x \right) e^{-G^2K/T} \right\}$$

(for the definition of used symbols see [5], (7.21)-(7.23)), uniformly for $T^\epsilon \leq G \leq T^{1/2-\epsilon}$. And finally we add the Ivic' estimate, [5], (7.62),

$$\int_{T-G}^{T+G} Z^2(t)\mathrm{d}t = \mathcal{O}(G\ln^2 T), \quad G \geq T^{1/3-\epsilon_0}, \quad \epsilon_0 = \frac{1}{108} \approx 0.009. \tag{1.7}$$

Remark 2. It is obvious that short intervals: $[T-G, T+G]$ with $G \in (0,1)$ are not included in the methods of Balasubramanian, Heath-Brown and Ivic leading to (1.5) and (1.6), (1.7).

In this work we present a new method how to deal with short parts ($1 \leq U \leq T^{1/3+\epsilon}$) and microscopic parts ($0 < U < 1$) of the Hardy-Littlewood integral

$$\int_T^{T+U} Z^2(t)\mathrm{d}t. \tag{1.8}$$

To attain this goal we will use only elementary geometric properties of Jacob's ladders.

Remark 3. Our new and elementary method leads to new expressions for the integrals of type (1.8) and these new expressions cannot be derived by means of methods of Balasubramanian, Heath-Brown and Ivic.

Remark 4. We will see that the Jacob's ladders (by means of the formula (1.3)) discover the feature of short and microscopic parts of the Hardy-Littlewood integral. From this point of view, the Jabob's ladders play the role of *Golem's shem* for the mentioned Hardy-Littlewood integral.

2. Tangent law

2.1. The basic idea is in the following theorem.

Theorem 1. *For* $0 < U < U_0 = T^{1/3+2\epsilon}$ *the following is true*

$$\begin{aligned} \int_T^{T+U} Z^2(t)\mathrm{d}t &= \left(U\ln\frac{\varphi(T)}{2} - aU \right)\tan[\alpha(T,U)] + \\ &+ \mathcal{O}\left(\frac{1}{T^{1/3-4\epsilon}}\right), \ a = \ln 2\pi - 1 - c, \end{aligned} \tag{2.1}$$

where $\alpha = \alpha(T,U)$ *is the angle of the chord of the curve* $y = 1/2\varphi(T)$ *that binds the points*

$$[T, \frac{1}{2}\varphi(T)], \ [T+U, \frac{1}{2}\varphi(T+U)]. \tag{2.2}$$

Proof. As a consequence of (1.3) we have

$$\begin{aligned}\int_T^{T+U} Z^2(t)\mathrm{d}t &= \frac{1}{2}[\varphi(T+U)-\varphi(T)]\left(\ln\frac{\varphi(T)}{2}-a\right)+ \\ &+ \mathcal{O}\left\{\frac{[\varphi(T+U)-\varphi(T)]^2}{T}\right\}+\mathcal{O}\left(\frac{\ln T}{T}\right).\end{aligned} \tag{2.3}$$

Using the equality (see [7], (5.2))

$$\frac{\varphi(T+U_0)-\varphi(T)}{T}=\mathcal{O}(1),$$

and (2.3) with $U=U_0$ and the Balasubramanian formula (1.5) we obtain the following inequality

$$\varphi(T+U)-\varphi(T)\le\varphi(T+U_0)-\varphi(T)=\mathcal{O}(T^{1/3+2\epsilon}),\quad 0<U\le U_0. \tag{2.4}$$

And now by making use of formulae (2.3) and (2.4) we have

$$\int_T^{T+U} Z^2(t)\mathrm{d}t=\frac{1}{2}[\varphi(T+U)-\varphi(T)]\ln\left(e^{-a}\frac{\varphi(T)}{2}\right)+\mathcal{O}\left(\frac{1}{T^{1/3-4\epsilon}}\right). \tag{2.5}$$

At this moment we need only to remind that

$$\frac{1}{2}[\varphi(T+U)-\varphi(T)]=U\tan\alpha,$$

where $\alpha=\alpha(T,U)$ is the angle of the chord binding the points (2.2), and therefore (2.1) follows from (2.5).

□

2.2. By a finer comparison of the formulae (1.5) and (2.1), with $U=U_0$, and by making use of

$$\frac{1}{\ln T}\ln\frac{\varphi(T)}{2}=1+\mathcal{O}\left(\frac{1}{\ln^2 T}\right),$$

(for this formula see [7], (6.6)) we can derive the following asymptotic formula

$$\tan[\alpha(T,U_0)]=1-\frac{1-c}{\ln T}+\mathcal{O}\left(\frac{1}{\ln^2 T}\right), \tag{2.6}$$

where $\alpha(T,U_0)$ is the angle of the chord binding the points

$$[T,\frac{1}{2}\varphi(T)],\ [T+U_0,\frac{1}{2}\varphi(T+U_0)]. \tag{2.7}$$

Remark 5. The asymptotic formula (2.6) is the geometric consequence of the Balasubramanian formula (1.5). The geometric consequence of the Ivic estimate (1.7) is the following estimate

$$\tan[\alpha(T,G)]=\mathcal{O}(\ln T).$$

2.3. Let us consider the set of all chords of the curve $y=\frac{1}{2}\varphi(T)$ which are parallel to the main chord that binds the points (2.7). Let the generic chord of this class bind the points $[N,\frac{1}{2}\varphi(N)]$ and $[M,\frac{1}{2}\varphi(M)]$. Then from (2.1) and (2.6) we obtain the following

Corollary 1. There is continuum of intervals $[N,M]\subset[T,T+T^{1/3+2\epsilon}]$ for which the following formula holds true

$$\begin{aligned} \int_N^M Z^2(t)\mathrm{d}t &= (M-N)\ln N+(2c-\ln 2\pi)(M-N)+ \\ &+ \mathcal{O}\left(\frac{M-N}{\ln T}\right)+\mathcal{O}\left(\frac{1}{T^{1/3-4\epsilon}}\right). \end{aligned} \tag{2.8}$$

Remark 6. Let us remark that the formula (2.8) cannot be obtained by methods of Balasubramanian, and estimates done by Heath-Brown and Ivic, since it holds true, for example, for system intervals of type: $[N,M]\subset(0,T^{1/6})$.

3. On microscopic parts of the Hardy-Littlewood integral in neighbourhoods of zeroes of the function $\zeta(\frac{1}{2}+iT)$

Let γ,γ' be a pair of neighbouring zeroes of the function $\zeta(\frac12+iT)$. The function $\frac12\varphi(T)$ is necessarily convex on some right neighbourhood of the point $T=\gamma$, and this function is necessarily concave on some left neighbourhood of the point $T=\gamma'$. Therefore, there exists the minimal value of $\rho\in(\gamma,\gamma')$ such that $[\rho,\frac12\varphi(\rho)]$ is the inflection point of the curve $y=\frac12\varphi(T)$. At this point, by properties of the Jacob's ladders, we have $\varphi'(\rho)>0$. Let furthermore $\beta=\beta(\gamma,\rho)$ be the angle of the chord binding the points

$$[\gamma,\frac12\varphi(\gamma)],\ [\rho,\frac12\varphi(\rho)]. \tag{3.1}$$

Then we obtain by our Theorem.

Corollary 2. For every sufficiently big zero $T=\gamma$ of the function $\zeta(\frac12+iT)$ the following formulae describing microscopic parts of the Hardy-Littlewood integral hold true

(A) the continuum of formulae

$$\begin{aligned} \int_\gamma^{\gamma+U} Z^2(t)\mathrm{d}t &= \left(U\ln\frac{\varphi(\gamma)}{2}-aU\right)\tan\alpha+\mathcal{O}\left(\frac{1}{\gamma^{1/3-4\epsilon}}\right), \\ & \alpha\in(0,\beta(\gamma,\rho)),\ U=U(\gamma,\alpha)\in(0,\rho-\gamma), \end{aligned} \tag{3.2}$$

where $\alpha=\alpha(\gamma,U)$ is the angle of the rotating chord binding the points $[\gamma,\frac12\varphi(\gamma)]$ and $[\gamma+U,\frac12\varphi(\gamma+U)]$,

(B) the continuum of formulae for chord parallel to the chord given by points (3.1)

$$\begin{aligned} \int_N^M Z^2(t)\mathrm{d}t &= \left[(M-N)\ln\frac{\varphi(N)}{2}-a(M-N)\right]\tan[\beta(\gamma,\rho)]+ \\ &+ \mathcal{O}\left(\frac{1}{\gamma^{1/3-4\epsilon}}\right),\quad \gamma<N<M<\rho. \end{aligned} \tag{3.3}$$

Remark 7. The notion *microscopic parts* of the Hardy-Littlewood integral has its natural origin in the following: by Karacuba's selbergian estimate (see [5], p. 265) we know that for *almost all* intervals $[\gamma,\gamma']\subset[T,T+T^{1/3+2\epsilon}]$ we have

$$\gamma'-\gamma<A\frac{\ln\ln T}{\ln T}\to 0,\ T\to\infty. \tag{3.4}$$

Remark 8. In connection with (3.4) we can remind that if the Riemann conjecture holds true then the Littlewood's estimate takes place

$$\gamma' - \gamma < \frac{A}{\ln\ln\gamma} \to 0,\ \gamma \to \infty,$$

(see [8], p. 296, simple consequence of the estimate $S(T) = \mathcal{O}(\ln T/\ln\ln T)$).

Remark 9. It is obvious that the formulae (3.2) and (3.3) for microscopic parts of the Hardy-Littlewood integral cannot be obtained from the Balasubramanian's formula or by estimates by Heath-Brown and Ivic.

4. Second class of formulae for parts of the Hardy-Littlewood integral beginning in zeroes of the function $\zeta(\frac{1}{2}+iT)$

Let $T = \gamma, \bar{\gamma}$ be a pair of zeroes of the function $\zeta(\frac{1}{2}+iT)$, where $\bar{\gamma}$ obeys the following conditions

$$\bar{\gamma} = \gamma + \gamma^{1/3+2\epsilon} + \Delta(\gamma),\ 0 \le \Delta(\gamma) = \mathcal{O}(\gamma^{1/4+\epsilon}),$$

(see the Hardy-Littlewood estimate for the distance between the neighbouring zeroes, [3], pp. 125, 177-184). Consequently,

$$U(\gamma) = \gamma^{1/3+2\epsilon} + \Delta(\gamma) \sim \gamma^{1/3+2\epsilon},\ \gamma \to \infty. \tag{4.1}$$

For the chord that binds the points

$$[\gamma, \frac{1}{2}\varphi(\gamma)],\ [\bar{\gamma}, \frac{1}{2}\varphi(\bar{\gamma})] \tag{4.2}$$

we have by (2.6) and (4.1)

$$\tan[\alpha(\gamma, U(\gamma))] = 1 - \frac{1-c}{\ln\gamma} + \mathcal{O}\left(\frac{1}{\ln^2\gamma}\right). \tag{4.3}$$

The continuous curve $y = \frac{1}{2}\varphi(T)$ lies below the chord given by points (4.2) on some right neighbourhood of the point $T = \gamma$ and this curve lies above that chord on some left neighbourhood of the point $T = \bar{\gamma}$. Therefore there exists a common point of the curve and the chord. Let $[\bar{\rho}, \frac{1}{2}\varphi(\bar{\rho})],\ \bar{\rho} \in (\gamma, \bar{\gamma})$ be such a common point that is the closest to the point $[\gamma, \frac{1}{2}\varphi(\gamma)]$. Then we obtain from our Theorem

Corollary 3. For every sufficiently big zero $T = \gamma$ of the function $\zeta(\frac{1}{2}+iT)$ we have the following formulae for the parts (1.8) of the Hardy-Littlewood integral

(A) continuum of formulae for the rotating chord

$$\begin{aligned} \int_\gamma^{\gamma+U} Z^2(t)\mathrm{d}t &= \left(U\ln\frac{\varphi(\gamma)}{2} - aU\right)\tan\alpha + \mathcal{O}\left(\frac{1}{\gamma^{1/3-4\epsilon}}\right), \\ & \tan\alpha \in [\eta, 1-\eta],\ U = U(\gamma,\alpha) \in (0, \bar{\rho}-\gamma), \end{aligned} \tag{4.4}$$

where $\alpha = \alpha(\gamma, U)$ is the angle of the rotating chord binding the points $[\gamma, \frac{1}{2}\varphi(\gamma)]$ and $[\gamma+U, \frac{1}{2}\varphi(\gamma+U)]$, and $0 < \eta$ is an arbitrarily small number,

(B) continuum of formulae for the chords parallel to the chord binding the points (4.2)

$$\begin{aligned} \int_N^M Z^2(t)\mathrm{d}t &= (M-N)\ln N + (2c - \ln 2\pi)(M-N) + \\ &+ \mathcal{O}\left(\frac{M-N}{\ln\gamma}\right) + \mathcal{O}\left(\frac{1}{\gamma^{1/3-4\epsilon}}\right),\ \gamma \le N < M \le \bar{\rho}. \end{aligned} \tag{4.5}$$

Remark 10. For example, in the case $\alpha = \pi/6$ we have from (4.4)

$$\int_{\gamma}^{\gamma+U(\pi/6)} Z^2(t)\mathrm{d}t = \frac{1}{\sqrt{3}}(U\ln\gamma - aU) + \mathcal{O}\left(\frac{U}{\ln\gamma}\right) + \mathcal{O}\left(\frac{1}{\gamma^{1/3-4\epsilon}}\right),$$

for every sufficiently big zero $T = \gamma$ of the function $\zeta(\frac{1}{2} + iT)$.

5. On I.M. Vinogradov's scepticism on possibilities of the method of trigonometric sums

5.1. I.M. Vinogradov, in the Introduction to his monograph [9], has analyzed the possibilities of the method of trigonometric sums (H. Weil' sums) in the problem of estimation of the remainder term $R(N)$ in the asymptotic formula

$$\pi(N) - \int_2^N \frac{\mathrm{d}x}{\ln(x)} = R(N) \tag{5.1}$$

(see [9], p. 13). He made the following remark in this:
Obviously, it is very hard to move on essentially in solution of the problem to find the order of the R term (willing to find $R = \mathcal{O}(N^{1-c}),\ c = 0.000001$) by making use of only some improvements of the H. Weyl's estimates and without making use of further important progresses in the theory of the zeta-function.

5.2. We will discuss in this section an analogue of the Vinogradov's scepticism in our case of estimation of the remainder term for the Hardy-Littlewood integral

$$\int_0^T Z^2(t)\mathrm{d}t - T\ln T - (2c - 1 - \ln 2\pi)T = R(T), \tag{5.2}$$

(analogue to (5.1). The first mathematician who applied the method of trigonometric sums to estimation of $R(T)$ was Titchmarsh (in 1934), and he received the result $R(T) = \mathcal{O}(T^{5/12+\epsilon})$, (see [8], p. 123). Balasubramanian then improved the Titchmarsh' result to $R(T) = \mathcal{O}(T^{1/3+\epsilon})$, [1]. The crucial result in this field obtained Good [2]: $R(T) = \Omega(T^{1/4})$.

Remark 11. It follows the Good's result that the estimate of type $R(T) = \mathcal{O}(T^{1/4+\epsilon})$ - which still represent an unbounded and unremovable absolute error in the formula for the Hardy-Littlewood integral (see Remark 1) - is the final result for the method of trigonometric sums in this question.

Remark 12. Since our almost exact formula (1.3) has been proved independently on the method of trigonometric sums, it, together with the Balasubramanian formula (1.4), can give a good keystone for an analogue of the I.M. Vinogradov's scepticism in the problem of possibility to find a finer representation of the Hardy-Littlewood integral $\int_0^T Z^2(t)\mathrm{d}t$.

5.3. I would like to describe my two main goals when I was working in Titchmarsh' sequences $\{Z(t_\nu)\}$, where $\{t_\nu\}$ is the Gramm's sequence. I wanted to

(a) improve the knowledge about the local variant of the classical Titchmarsh formulae (see [3], pp. 221, 222)

$$\sum_{\nu=\nu_0}^{N} Z(t_{2\nu}) = 2N + \mathcal{O}(N^{3/4}\ln^{3/4} N),$$
$$\sum_{\nu=\nu_0}^{N} Z(t_{2\nu+1}) = -2N + \mathcal{O}(N^{3/4}\ln^{3/4} N),$$

(b) prove mean-value theorems for the function $Z(t)$ on related non-connected sets.

To solve the tasks (a) and (b) I have first used the method of trigonometric sums (see [5], p. 260, 265, 266, [4], p. 37). In the task (a) I have improved the Titchmarsh exponent as $3/4 \to 1/6$, i.e. I improved that exponent by 77.7%.

In the task (b) I obtained a new class of mean-value theorems (see [6]), corresponding to the exponent 1/6

$$\frac{1}{m\{G_1(x)\}}\int_{G_1(x)} Z(t)\mathrm{d}t \sim 2\frac{\sin(x)}{x},$$
$$\frac{1}{m\{G_2(y)\}}\int_{G_2(y)} Z(t)\mathrm{d}t \sim -2\frac{\sin(y)}{y}, \quad T\to\infty.$$

In connection with the task (a) I have also essentially improved the Hardy-Littlewood exponent 1/4 (since 1918) to 1/6 (problem of estimation of distance of neighbouring zeroes of the function $\zeta(\frac{1}{2}+iT)$, see [3], p. 125, 177-184).

Let us follow the sequence of improvements:

- Moser - 33.3% improvement of the Hardy-Littlewood exponent 1/4
- Karacuba - 6.25% improvement of the exponent 1/6
- Ivic - 0.19% improvement of the Karacuba's exponent.

Remark 13. The sequence of improvements of the exponent of type $0.19\%, \dots$ show that scepticism of I.M. Vinogradov takes place also in the possibility of successful application of the method of trigonometric sums in the problem of crucial improvement of the exponent 1/6.

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